DOCUMENT RESUME

ED 441 670 SE 063 544

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TITLE Do Theories of Conceptual Change Explain the Difficulties of Enlarging the Number Concept in Mathematics Learning?
PUB DATE 2000-04-00
NOTE 8p.; Paper presented at the Annual Meeting of the American Educational Research Association (New Orleans, LA, April 24-28, 2000).
PUB TYPE Reports - Research (143) -- Speeches/Meeting Papers (150)
EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS Calculus; *Concept Formation; Foreign Countries; Higher Education; Learning Processes; Learning Theories; *Mathematicians; Mathematics Education; *Number Concepts; Secondary Education
IDENTIFIERS *Finland

ABSTRACT

This paper discusses the theories of conceptual change and how they explain the difficulties in mathematics learning, especially in enlarging the number concept. Two studies are presented, one of which examines the research questions: What is the role of prior knowledge in students' answers to questions about the density of rational and real numbers? and Do theories of conceptual change explain the difficulties students have in learning these concepts? Ten university professors of mathematics were interviewed about their personal learning histories as professional mathematicians, and were asked what they remembered about the initial learning of the real number concept and how they currently conceptualize the numbers system. The main result from the interviews was that most of the professors remembered that learning to understand the notion of real numbers was something unique, and it required them to move into a new kind of abstract thinking, free from the constraints of prior mathematical intuition. In the second study a number concept test was administered to 640 calculus students (aged 17-18 years) from 24 randomly selected Finnish upper secondary schools in the usual classroom conditions. The test included identification, classification, and construction problems in the domain of rational and real numbers. The role of the students' prior knowledge was then analyzed using their answers to four critical questions. Results from the second study indicate that the process of knowledge acquisition, especially in mathematics, involves restructuring as well as enriching one's prior knowledge structures, which can take place at many different levels. (Contains 30 references.) (ASK)

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Do Theories of Conceptual Change Explain the Difficulties of Enlarging the Number Concept in Mathematics Learning?

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Handout for the poster presented in AERA annual meeting 24-28th April, New Orleans, USA

DO THEORIES OF CONCEPTUAL CHANGE EXPLAIN THE DIFFICULTIES OF ENLARGING THE NUMBER CONCEPT IN MATHEMATICS LEARNING?

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INTRODUCTION

Mathematics is supposed to have a completely hierarchical structure in which all new concepts logically follow from prior ones (Dantzig 1954). In the process of enlarging the number concept, the set of natural numbers is embedded in the set of real numbers or even "thrown out" and replaced by corresponding more advanced numbers (Landau 1960). The cognitive processes of concept acquisition, however, seem not to follow the mathematical logic (Dreyfus 1991). Although these concepts were used and pondered early on by Democritus (Boyer 1959) it was not until the end of the nineteenth century that mathematicians formed the rigorous definitions of numbers, and they were defined in reverse order compared to the teaching order of the ordinary curriculum (Kline 1980). It is possible that the harmonious whole and logical structure of mathematics appears as logical and continuous only for the experts of mathematics but as fragmentary and discontinuous for the student who is struggling to understand (Lehtinen 1998).

THEORIES OF CONCEPTUAL CHANGE

Theories of conceptual change are based on empirical studies of learning scientific concepts. These theories highlight the relationship between **prior knowledge** and the information to be learned as one of the most crucial factors in determining the quality of learning. The new-nativistic theories (Spelke 1991; Gallistel & Gelman 1992; Carey & Spelke 1994; Gelman & Brenneman 1994) argue that human reasoning is guided by a collection of **innate domain-specific systems** of knowledge. The theory of **ontological categories** (Chi, Slotta & deLeeuw 1994; Chi & Slotta 1994; Ferrari and Chi 1998) is based on the philosophical analysis of ontological categories of the prior knowledge. The naïve framework theory (Vosniadou 1994, Vosniadou & Ioannides 1998) explains that children's **naïve framework theory of physics** consist of ontological and epistemological presumptions about the kinds of entities we assume to exist and the way they are categorized.

These theories describe two levels of difficulty in the learning process. The easier level of conceptual change means *enrichment* of one's prior knowledge structure. In this case the prior knowledge is sufficient for accepting the new specific information. The more difficult conceptual change is needed when the prior knowledge is not sufficient for the new information but needs *revision*. (Vosniadou 1994.) Sometimes students do not see or understand the reason for changing their prior knowledge and logic even though revision would be necessary. They then attempt to synthesize scientific concepts with their naïve beliefs. (Vosniadou & Ioannides 1998.)

The theory of **phenomenological primitives** (diSessa 1993) describes the prior intuitive knowledge as composed of pieces of small knowledge structures which typically are self-explanatory beliefs describing the causality of a phenomena. Learning in this frame of reference means reorganizing the intuitive knowledge where recognition occurs as layers affected by other previously activated elements. The drastic revision in the intuitive knowledge is the change in function of the primitives. They cease to be self-explanatory and change to more complex structures with scientific explanations.

WHY THE THEORIES OF CONCEPTUAL CHANGE?

Although the theories of conceptual change have earlier been used in the field of science learning, it seems reasonable to suggest that these theories also explain the difficulties in the learning of mathematics, especially in enlarging the number concept. There are at least four observations that seem to refer to that

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direction: **First**, the mathematical conceptions are of dual nature, operational and structural, where operational conception naturally precedes the structural and where there is a deep ontological gap between them (Sfard 1991; Sfard & Linchevski 1994). **Second**, in the history of mathematics there was a long development period between the operational use and the structural formalization of these concepts and the long and difficult journey in the formalization process (Boyer 1994; Kline 1980; Dantzig 1957). **Third**, in the concepts of advanced mathematics there is a high level of abstraction and complexity (Dreyfus 199; Tall 1991) but in the mathematics of the students everyday life a comparative low nature of abstraction (Hatano 1996). **Fourth**, every extension to the number concept demands accepting new specific knowledge but also new logic that more or less contradicts the prior fundamental logic of natural numbers (Russell 1993; Kieren 1992; Hartnett & Gelman 1998; Stafilidou & Vosniadou 1999). Therefore misconceptions and insufficient learning are possible at every extension. The process of conceptual change in enlarging the number concept to the domain of real numbers is an especially radical one. This process (at high school level) involves, for example understanding the hierarchical nature of real numbers where rational numbers, integers and natural numbers are as sub sets. But in addition to that, it involves understanding the mathematical concepts of limit and continuity. These concepts have a long and difficult history (Boyer 1959, Kline 1980) and they are difficult for the students today (Lehtinen, Merenluoto & Kasanen 1997; Kaput 1994; Fischbein, Jehiam & Cohen 1995; Tall & Vinner 1981; Fischbein, Tirosh & Hess 1979; Schwarzenberger & Tall 1978).

RESEARCH QUESTIONS

1. What is the role of prior knowledge in students' answers to questions pertaining the density of rational and real numbers?
2. Do the theories of conceptual change explain the difficulties the students have in learning these concepts?

STUDY 1

Method

We interviewed ten university professors of mathematics about the personal learning histories as professional mathematicians and asked them to tell us what they remembered about the initial learning of the real number concept and how they currently conceptualise the system of numbers.

Results

The main result **from the interviews** was that most of the professors remembered that learning to understand the notion of real numbers was something unique and it required them to move into a new kind of abstract thinking, free from the constraints of prior mathematical intuition.

STUDY 2

Method

A number concept test was administered to 640 (first measurement) calculus students (age 17-18 years), from 24 randomly selected Finnish upper secondary schools in the usual classroom conditions. The test included identification, classification and construction problems in the domain of rational and real numbers. While giving their answers to the questions the students were also asked to estimate their sense of certainty on a scale from 1 to 5, where 1 means that they did not know and the answer was a guess and 5 means that they are as absolutely sure as they know that 1+1=2. Another number concept test with parallel questions was administered six months later to 272 (second measurement) of the students in the previous test.

Results

In this report we analyse the role of the students' prior knowledge by using their answers to four critical questions. In all these questions the students were asked to explain the concept with their own words. These were the parallel questions pertaining to the 'compact' nature of rational and real numbers. In the context of the number line the situation is described with the word 'density' and limit and in the context of function the

parallel concepts are described with words 'continuity' and limit. (The concept of limit is traditionally taught in the context of functions and the continuity of functions.)

1. The density and limit on the number line

In one of the questions pertaining to the concept of the number line the students were asked *"How many fractions are there between two given fractions on the number line?"* Of the given answers 45.8% were based on the 'add one' logic of natural numbers (figure 1). The recognition of infinite density of the number line was seen in 24.1% of given answers and 24.3% explained this density by refering to the possibility of increasing the density of the number line by adding decimals or changing the denominator etc. These were operational level answers (Sfard 1991). Only 5.7% of the students showed some structural abstraction of the concept in their explanations of infinite amount of fractions between any two given fractions.

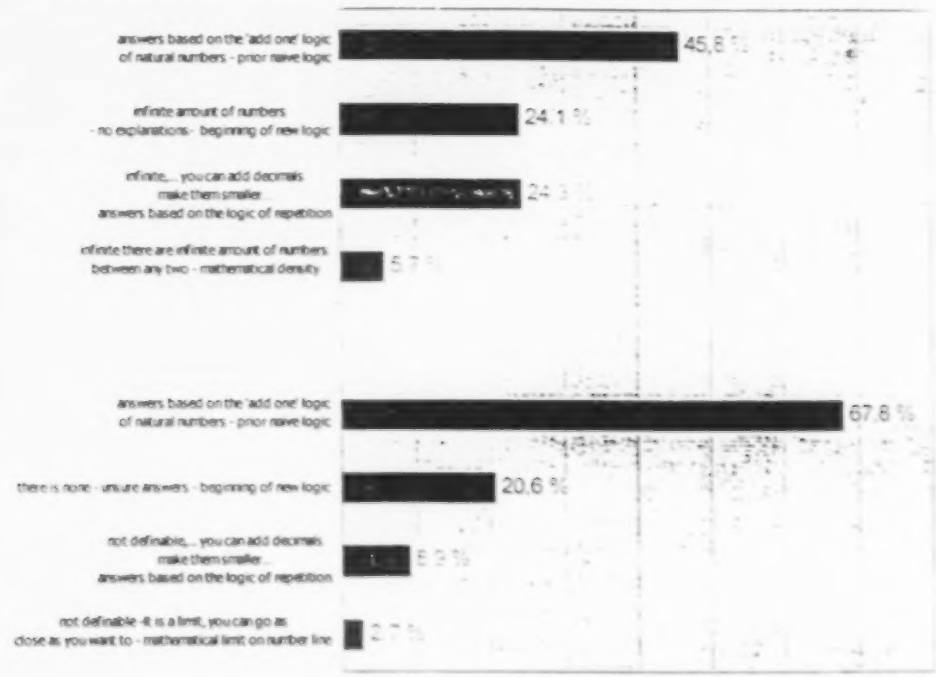
answers based on the 'add one' logic of natural numbers - prior naive logic
45,8 %
infinite amount of numbers - no explanations - beginning of new logic
24,1 %
infinite,... you can add decimals make them smaller... answers based on the logic of repetition
24,3 %
infinite there are infinite amount of numbers between any two - mathematical density
5,7 %
answers based on the 'add one' logic of natural numbers - prior naive logic
67,8 %
there is none - unsure answers - beginning of new logic
20,6 %
not definable,... you can add decimals make them smaller... answers based on the logic of repetition
8,9 %
not definable -it is a limit, you can go as close as you want to - mathematical limit on number line
2,7 %

Figure 1. Percentages of students' answers to questions pertaining to the (1) density of the number line (*how many fractions between two given fractions?*) and (2) limit on the number line (*which one is the 'next'?*) The corresponding percentages from the second measurement were in the question (1) 48.7%, 19.5%, 25.4% and 6.4% and in the question (2) respectively 73.5%, 13.0%, 10.9% and 2.6%.

The distribution of the answers given to the second question: *Which fraction comes next after 3/5?* was statistically significantly different (χ^2 = 155.52, p< .001) in comparision with the previous question. Intuition of the 'next' number seems so obvious and certain that 67.8% of those who answered the questions gave the number 4/5. One fifth (20.6%) gave indefinite, uncertain answers: "there is none" or "3/5 + 1/∞". Only 11.6% of given answers were mathematically correct: "it is not possible to define the 'next'". Only 2.7% of given answers were based on correct structural knowledge about rational numbers: it is a limit, one can approach arbitrary close.

The majority of the students relied on their **prior knowledge and their beliefs of numbers**, which seemed to be composed of the following beliefs: 1) there is always a next object, 2) there is an 1-1 correspondence between numbers and objects, 3) there must always be a next number and 4) the number line is continued by repeating the action of adding one. The majority of those who identified the infinite density of number line

based their answers on the logic of repetition. This level of comprehension of the infinite density seems to be possible by merely enriching the prior knowledge. Whereas those answers where some structural abstraction was identified indicated that these students had made some reconstruction or revision of their prior knowledge. A radical revision in the prior knowledge is needed to comprehend that it is not possible to define the next number in the domain of rational and real numbers.

In the first question 54.2% of the given answers were at least on the recognition level whereas in the second question only 32.2% showed at least some kind of recognition of the infinite divisibility. Moreover, the mean of the estimated certainty in the primitive ("add one") level of answers was significantly higher ($F=10.45$, $p=.0014$) in the answers to the second guestion than in the first question. These results refer to the obvious intuition of always having the 'next' number. The difference between these questions was seen also in the percentages of students who did answer the question:in the first measurement 79.2% of the students answered to the first question and 57.8% to the second, the percentages in the second measurement were 87.1% and 84.6% respectively.

2. The continuity and limit of a function

The concept of limit is fundamental to real numbers, but it is traditionally taught in the context of functions, where continuity is the subconcept of the limit. That is why we asked the students to also describe the concepts of continuity and limit of functions with their own words.

In both the cases of continuity and limit of a function the primitive level of answers gave implications of a different kind of prior knowledge compared with the answers pertaining to the questions on the number line. The majority of the students relied on their **prior intuitive knowledge,** which seemed to be composed of self explanatory beliefs giving the cause of continuity and the mechanism of the limit (figure 2). The students described the continuity caused by motion: something is continuous if it does not stop (13.3% of given answers) or something that does not break or "jump" or is possible to draw without lifting a pen (27.9%), or they tried to describe the continuity as something without 'gaps' when they answered that something is continuous if it is defined at every x (42.5%). In all of these answers the students seemed to base their answers on their enriched prior knowledge or everyday experiences of continuity caused by motion or having 'no gaps'. Only 4.9% of the answers showed hints of radical reconstruction of the prior knowledge. In these the students gave mathematically correct answers of continuity as a static state of affairs based on the concept of limit.

In the case of continuity the prior everyday concept of the phenomena seems to support the development of the concept. In the case of the concept of limit, however, 48.9% of the answers given were based on the mechanism of limit as a 'limitor', giving the implication of the students prior knowledge and experience of limits as as stopper of continuity. The role of limit as a 'stop-sign' was there although more implicitly when the students described limits as something you can approach but never reach (30.5% of given answers) and also in the little more defined (enriched) answers where they described limits as a value the function approaches when the variable approaches some other value (15.6%). These answers, where the students described the limit as something "the function approaches but does not (ever) reach" indicate intuitive understanding of the abstraction of the limit concept. In these answers also the dynamic motion of approaching was identified. A drastic revision is needed, however, before one is are able to describe the limit as the static state of affairs. Hints of this kind of situation was seen only in 4.96% of the given answers.

Although significant progress in the second measurement happened only in the answers pertaining to the continuity of a function ($\chi^2=23.86$, $p=.000$), it still seems difficult to capture the understanding that the existence of limit is prerequisite to continuity. This difficulty seems at least partly be due to the primitive intuition where limit stops the continuity. The concept of limit is very complicated and abstract and seems to be very difficult to the students. These results indicate, that the difficulty of the concept is not only due to the complexity but also to the quality of the prior knowledge of the students where the primitive intuitive knowledge is contradictive with the scientific concept.

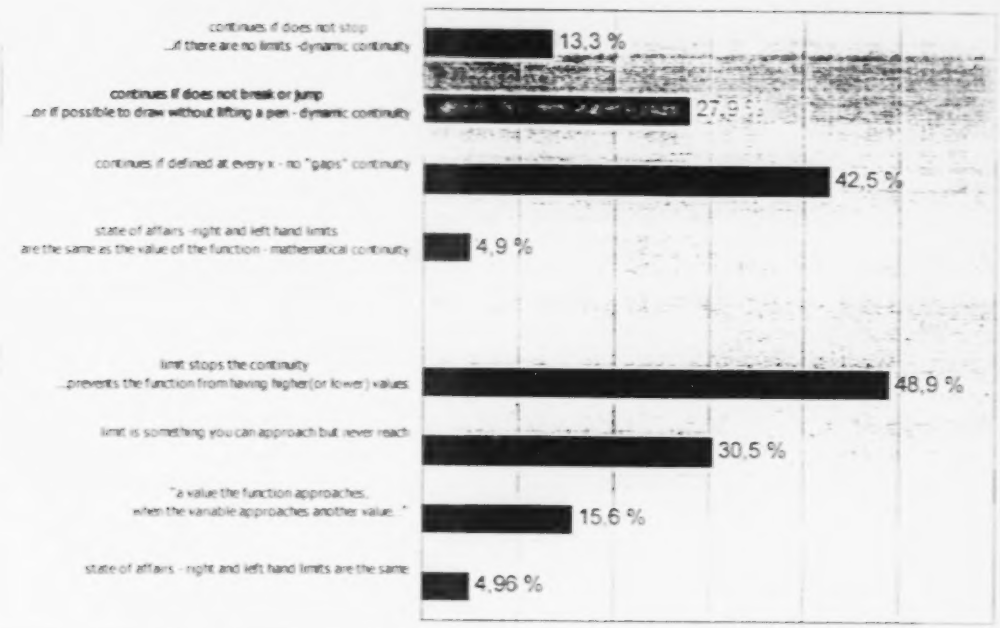
continues if does not stop
...if there are no limits -dynamic continuity
13,3 %
continues if does not break or jump
...or if possible to draw without lifting a pen - dynamic continuity
27,9 %
continues if defined at every x - no "gaps" continuity
42,5 %
state of affairs -right and left hand limits
are the same as the value of the function - mathematical continuity
4,9 %
limit stops the continuity
...prevents the function from having higher(or lower) values
48,9 %
limit is something you can approach but never reach
30,5 %
"a value the function approaches,
when the variable approaches another value..."
15,6 %
state of affairs - right and left hand limits are the same
4,96 %

Figure 2. Percentages of students' answers to questions pertaining to the (1) definition of continuity with their own words and (2) definition of limit with their own words. The corresponding percentages from the second measurement were in the question of continuity 6.1%, 38.0%, 43.7% and 12.2% and in the question of limit respectively 47.7%, 35.5%, 10.2% and 6.6%.

CONCLUSION

The results from the study 2 indicate that the process of knowledge acquisition, especially in mathematics, involves restructuring as well as enrichment of one's prior knowledge structures which can take place at many different levels. The concepts which are seemingly possible to learn by only enriching the prior knowledge without making a revision seem be easier to the students. These findings support the two level of difficulty defined by the theories of conceptual change and also the strong and sometimes restrictive nature of the prior knowledge. The results refer also to a possibility where moderate enrichment kind of conceptual change might act as an obstacle preventing the more advanced understanding. The students may not even see or understand the need for revision.

The majority of the students had not restructured their prior system of beliefs and logic of natural numbers but were just in the beginning of the process. Many of them spontaneously used the logic of natural numbers in the domain of rational numbers. Most of them had, nevertheless, isolated facts of the more advanced numbers in their number concept. Less than 15 % of the students, however, consistently used the logic and operational knowledge of rational and real numbers. In the results form the test six months later the percentages of advanced answers were a slightly higher but the percentages of primitive answers were significantly higher than in the previous test. The concepts where radical revision in the prior knowledge is needed seem quite resistant to the traditional teaching. The majority of the students seem not to b free from the constraints of prior mathematical intuition and may be not even aware of the need to change the underlying logic of their thinking.

EDUCATIONAL AND SCIENTIFIC IMPORTANCE OF THE STUDY

These results seem to refer firstly to the importance of the need to be aware of the changed logic on the level of more advanced numbers. Secondly these results refer to the implication of a better learning environment for these abstract concepts. Early development of numbers is dealt with thoroughly in learning research but there is comparatively little research dealing with later extensions of the number concept. These findings

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suggest important theoretical considerations for planning learning environments which better support the process of conceptual changes of the students.

REFERENCES

Boyer, C. B. 1959. The history of calculus and its conceptual development. (Orig. print 1949) New York: Dover Publications.

Carey, S. & Spelke, E. 1994. Domain-specific knowledge and conceptual change. In Hirsfeld, L. A. & Gelman, S. A. (eds.) Mapping the mind. Domain specificy in cognition and culture. Cambridge, MA: Cambridge University Press, 169-200.

Chi, M. T. H., Slotta, J. D. & de Leeuw, N. 1994. From things to process: A theory of conceptual change for learning science concepts. Learning and Instruction, 4, 27-43.

Chi, M. T. H. & Slotta, J. D. 1993. The Ontological coherence of intuitive physics. Cognition and Instruction. 10(2&3), 249-260.

Comu, B. 1991. Limits, In Tall, D. (ed.) Advanced mathematical thinking. Dordrecht: Kluwer Academic Publishers,153 – 166.

Dantzig, T. 1954. Number, the language of science. (Orig.print 1930.) New York: The Free Press.

Dreyfus, T. 1991. Advanced mathematical thinking processes. In Tall, D. (ed.) Advanced mathematical thinking. Dordrecht: Kluwer Academic Publishers, 25-41.

Fishbein, E., Jehiam, R. & Cohen, D. 1995. The concept of irrational numbers in high-school students and prospective teachers. Educational Studies in Mathematics, 29, 29 - 44.

Fishbein E., Tirosh, D. & Hess, P. 1979. The intuition of infinity. Educational Studies in Mathematics. Cambridge: Reidel Publishing Company, 10, 3-40.

Ferrari, M. & Chi, M. T. H. 1998. The nature of naive explanations of natural selection. In press: To appear in the International Journal of Science Education.

Gallistel, G. R. & Gelman, R. 1992. Preverbal and verbal counting and computation. Cognition, 44, 43-74.

Gelman, R. & Brenneman, K. 1994. First principles can support both universal and culture-specific learning about number and music. In Hirshfeld, L. & Gelman, S. (eds.) Mapping the mind. Domain specificy in cognition and culture. Cambridge: Cambridge university press, 369- 390.

Hartnett, P. & Gelman, R. 1998. Early understanding of numbers: paths or barriers to the construction of new understandings? Learning and Instruction, Vol. 8, No 4, 341-374.

Hatano, 1996. A conception of knowledge acquisition and its implications for mathematics education. In Steffe, P., Nesher, P., Cobb, P., Goldin, G. & Greer, B. (eds.) Theories of mathematical learning, New Jersey: Lawrence Erlbaum, 197 - 217.

Kaput, J. J. 1994. Democratizing access to calculus: new routes to old roots. In Schoenfeld, A. J. (ed.)Mathematical thinking and problem solving. Hillsdale, New Jersey: Lawrence Erlbaum publishers, 77-155.

Kieren, T. 1992. Rational and fractional numbers as mathematical and personal knowledge: implications for curriculum and instruction. In Leinhardt, G, Putnam, R. & Hattrup, R. (eds.) Analysis of arithmetic for mathematics teaching. New Jersey: Lawrence Erlbaum, 323-371.

Kline, M. 1980. Mathematics. The loss of certainty. New York: Oxford University Press.

Landau, E. 1960. Foundations of analysis. The arithmetic of whole, rational, irrational and complex numbers. New York: Chelsea Publishing. (Orig print 1951)

Lehtinen, E. 1998. Conceptual change in mathematics, paper presented in Second European Symposium on Conceptual Change. Madrid.

Lehtinen, E., Merenluoto, K. & Kasanen, E 1997. Conceptual change in mathematics: From rational to (un)real numbers. European Journal of Psychology of Education. Vol. XII, (2), 131-145.

diSessa, A. 1993. Toward an epistemology of physics. Cognition and instruction, 10(2&3), 105-225.

Sfard, A. 1991. On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics, 22(1), 1-36.

Sfard, A & Linchevski, L. 1994. The gains and pitfalls of reification - the case of algebra. Educational studies in Mathematics, 26, 191-228.

Schwarzenberger, R. L.E. & Tall, D. O. 1978. Conflict in the learning of real numbers and limits. Mathematics Teaching 82, 44-49.

Spelke, S. E. 1991. Physical knowledge in infancy: Reflections on Piaget's theory. In Carey, S. & Gelman, R. (eds.) The epigenesis of mind: Essays on biology and cognition. Hillsdale, NJ: Erlbaum, 133-170.

Stafilidou, M. & Vosniadou, S. 1999. Children's believes about the mathematical concept of fraction. Abstracts. Proceedings of the 8th European Conference for Research on Learning and Instruction, Biennial Meeting, August 24-28, Göteborg, Sweden, 478.

Tall, D. & Vinner, S. 1981. Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational Studies in Mathematics, 12, 151-169.

Tall, D. 1991. The psychology of advanced mathematical thinking. In Tall, D. (ed.) Advanced mathematical thinking. Dordrecht: Kluwer Academic Publishers, 3-24.

Vosniadou, S. 1994. Universal and culture-specific properties of children's mental models of the earth. In Hirsfeld, L. A. & Gelman, S. A. (ed.) Mapping the mind. Domain specificity in cognition and culture. Cambridge, MA: Cambridge University Press, 412-430.

Vosniadou, S. & Ioannides, C. 1998. From conceptual development to science education: a psychological point of view. International Journal of Science Education, (20) 10, 1213 – 1230.